Wicked Waves - Solution

November 15, 2023

Difficulty: ★ ☆ ☆ ☆ ☆
Key words: Calculus

We can find the x value where the maximum occurs by differentiating the function and setting equal to zero. This gives:

$$\frac{\partial h}{\partial x} = n\sin(x)^{n-1}\cos(x)^{m+1} - m\sin(x)^{n+1}\cos(x)^{m-1} \quad = n\cos(x)^2 - m\sin(x)^2. \tag{1}$$

Equating to zero and solving provides the solution:

$$x^* = \arctan(\pm\sqrt{n/m}). \tag{2}$$

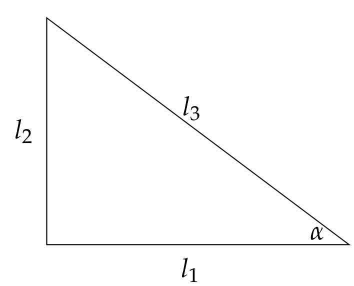

Figure 1: Triangle with $\pi/2$ angle.

Now we can find the maxima of the function by plugging in these x^* values in the function. First we need some identities to work everything out smoothly. Using Figure 1 we can see that:

$$\alpha = \arctan(\frac{l_2}{l_2}), \text{ and} \tag{3}$$

$$\sin(\alpha) = \frac{l_2}{l_3}. \tag{4}$$

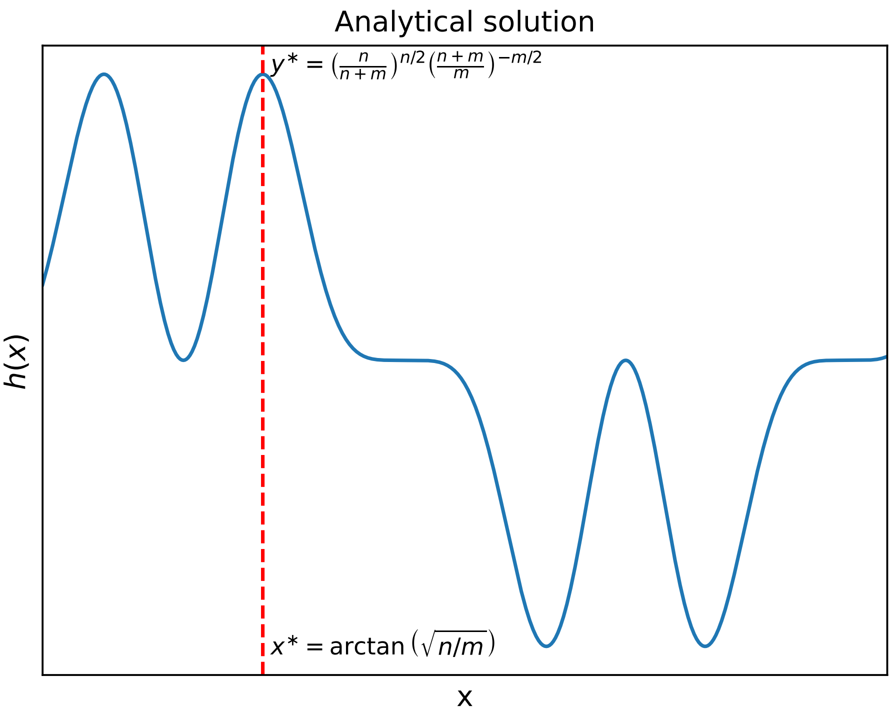

Figure 2: Function $h(x)$ for $n = 2$, $m = 5$.

By setting $l_1 = \sqrt{m}$ and $l_2 = \sqrt{n}$, we get:

$$\sin\left(\arctan\left(\frac{\sqrt{n}}{\sqrt{m}}\right)\right) = \frac{l_2}{l_3} = \frac{\sqrt{n}}{\sqrt{n+m}}. \tag{5}$$

We can do the same routine for the cosine. This will give us:

$$\cos\left(\arctan\left(\frac{\sqrt{n}}{\sqrt{m}}\right)\right) = \frac{l_2}{l_3} = \frac{\sqrt{m}}{\sqrt{n+m}}. \tag{6}$$

Combining everything, we end up with the expression:

$$y^* = \left(\frac{n}{n+m}\right)^{n/2}\left(\frac{m}{n+m}\right)^{m/2}.$$